On portfolio diversification; A case study

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A common belief: adding extra asset to a portfolio will automatically reduce the portfolio risk. We provide a counter-example resorting only to the simplest algebra and explain why this erroneous belief is so common.

I. DIVERSIFICATION OF A PORTFOLIO WITH EQUAL AND EQUALLY CORRELATED RETURNS

Consider a portfolio of N assets. Variance of the return, r_n, of the n-th asset reads:

$$\sigma_n^2 = \langle r_n - \langle r_n \rangle \rangle^2. \quad (1)$$

By $\langle A \rangle$ we denote the mean/expected value of an observable A.

Assets are in general correlated and degree of correlation is described in terms of the correlation coefficient $\rho = \sigma_{mn}/\sigma_m^2\sigma_n^2$, with covariance being:

$$\sigma_{mn} = \langle (r_n - \langle r_n \rangle)(r_n - \langle r_n \rangle) \rangle. \quad (2)$$

Variance of a equally weighted portfolio of assets with equal variances ($\sigma_m^2 = \sigma^2$) and correlations ($\sigma_{mn} = \rho\sigma^2$) is:

$$\sigma_P^2 = \sum_{m=0}^{N} \sum_{n=0}^{N} u_m u_n \sigma_{mn}, \quad (3)$$

u_n is the proportion of the total investment in the asset n. We consider long positions only, hence we require that all the weights are non-negative $u_n \geq 0$ and they sum to 1:

$$\sum_{n=1}^{N} u_n = 1. \quad (4)$$

In a equally weighted portfolio: $u_n = 1/N$, N being number of assets, the portfolio variance reads:

$$\sigma_P^2 = \sum_{m=0}^{N} \sum_{n=0}^{N} u_m u_n \sigma_{mn} = \quad (5)$$

$$= \sum_{m=0}^{N} \sum_{n=0}^{N} \frac{1}{N^2} \sigma_{mn} \quad (6)$$

$$= \frac{1}{N^2} \left[\sum_{n} \sigma_{nn} + \sum_{m \neq n} \sigma_{mn} \right] = \quad (7)$$

$$= \frac{1}{N^2} \left[N\sigma^2 + N(N-1)\rho\sigma^2 \right] = \quad (8)$$

$$= \frac{\sigma^2}{N} \left[1 + (N-1)\rho \right]. \quad (9)$$

(We inserted $\sigma_{nn} = \sigma^2, \sigma_{mn} = \rho\sigma^2$.) For σ, ρ being constant, the portfolio variance reduces by adding an additional asset, since σ_P is a monotonically decreasing function of the number of assets N.

Expected value of return of an equally weighted N-asset portfolio reads:

$$\langle r_P \rangle_N = \sum_{n=1}^{N} u_n r_n = \frac{1}{N} \sum_{n=1}^{N} r_n. \quad (10)$$

When we add an asset, the portfolio return reads:

$$\langle r_P \rangle_{N+1} = \frac{1}{(N+1)} \sum_{n=1}^{N+1} r_n$$

$$= \frac{r_{N+1}}{(N+1)} + \frac{N}{(N+1)} \langle r_P \rangle_N \quad (11)$$

$$= \frac{r_{N+1}}{(N+1)} + \frac{1}{(1+1/N)} \langle r_P \rangle_N. \quad (12)$$

By adding an asset to the portfolio the portfolio return increases, $\langle r_P \rangle_{N+1} > \langle r_P \rangle_N$, iff $r_{N+1} > \langle r_P \rangle_N$.

In the following section we demonstrate that, in some cases, adding an asset to a portfolio can increase the portfolio variance.

II. PORTFOLIO OF THREE ASSETS WITH UNEQUAL COVARIANCES

Consider a portfolio of three assets. For simplicity assume that returns of all three assets have same variance $\sigma_n^2 = \sigma^2$. Correlation coefficient of returns of the first and the second asset is ρ_0. Correlation coefficient of returns of the first and the third and the second and the third asset is $\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = \rho\sigma^2$. When working with multiple assets it is convenient to consider the covariance matrix:

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho_0 & \rho \\ \rho_0 & 1 & \rho \\ \rho & \rho & 1. \end{pmatrix} \quad (13)$$

We want to find positive weights (long positions only) $u_1 \geq 0, u_2 \geq 0, u_3 = 1 - u_1 - u_2 \geq 0$ that minimize the portfolio variance:

$$\sigma_P^2 = \sum_{m=1}^{3} u_m^2 \sigma^2 + \sigma^2 \sum_{m \neq n} u_m u_n \Sigma_{mn} =$$

$$= \sigma^2[(u_1^2 + u_2^2 + u_3^2) + 2u_1 u_2 \rho_0 + 2(u_1 + u_2)u_3\rho]. \quad (14)$$

We need to solve a constrained minimization problem.

Since portfolio variance is a quadratic form, it can be conveniantly written using a matrix/vector notation:

$\sigma_P^2 = \mathbf{u}^T\Sigma\mathbf{u}$, $\mathbf{u}$ is a vector of asset weights: $\mathbf{u} = (u_1, u_2, \ldots, u_N)$. In our example $\mathbf{u} = (u_1, u_2, 1-u_1-u_2)$.

We consider two cases. In the first case we assume that first and the second asssets are completely uncorrelated, $\rho_0 = 0$, while in the second case we assume the correlation is finite $\rho_0 \neq 0$.

A. Case I: $\rho_0 = 0$

When first and the second asset are uncorrelated, $\rho_0 = 0$, our covariance matrix becomes:

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & 0 & \rho \\ 0 & 1 & \rho \\ \rho & \rho & 1. \end{pmatrix} \tag{15}$$

Hence, the portfolio variance reads:

$$\sigma_P^2 = \mathbf{u}^T\Sigma\mathbf{u} \tag{16}$$
$$= \sigma^2[u_1^2 + u_2^2 + 2\rho(u_1+u_2)(1-u_1-u_2)$$
$$+ (1-u_1-u_2)^2]. \tag{17}$$

We want to minimize the portfolio variance with the following constraints:

$$u_1 \geq 0, \tag{18}$$
$$u_2 \geq 0, \tag{19}$$
$$u_1 + u_2 \leq 1, \tag{20}$$
$$\sigma_P^2 \geq 0. \tag{21}$$

First three constraints follow from the long position only requirements.

To find a solution we employ miminization using Lagrange multipliers and Kuhn-Tucker conditions and arrive to the following set of equations:

$$u_1\frac{\partial L}{\partial u_1} = 0, \tag{22}$$
$$u_2\frac{\partial L}{\partial u_2} = 0, \tag{23}$$
$$\lambda\frac{\partial L}{\partial \lambda} = 0. \tag{24}$$

The Lagrange equation reads:

$$L = -\sigma_P^2 + \lambda(1-u_1-u_2). \tag{25}$$

We reduced the minimization problem to a system of three equations. This system has seven solutions for u_1, u_2, λ, however only two of them have $u_1 > 0, u_2 > 0$ (in the other 5 solutions either u_1, either u_2 are zero).

The first solution is:

$$u_1 = \frac{1}{2}, \tag{26}$$
$$u_2 = \frac{1}{2}, \tag{27}$$
$$u_3 = 0, \tag{28}$$
$$\lambda = 2\rho - 1, \tag{29}$$
$$s_1 = \sigma_P^2 = 1/2. \tag{30}$$

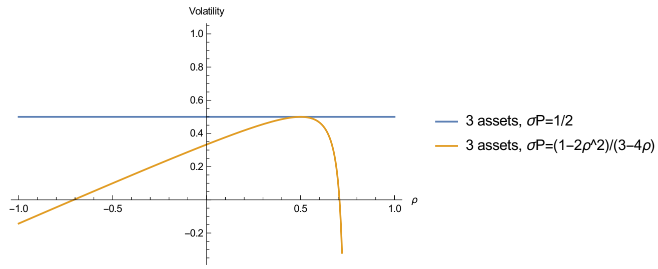

Figure 1. Portfolio volatility for two cases, $s_1 = \sigma_p^2 = 1/2, s_2 = \sigma_P^2 = (1-2\rho^2)/(3-4\rho)$. When correlations are strong, adding a third asset increases the portfolio volatility. Results when $s_2 < 0, \rho < -3/4$ are unrealistic.

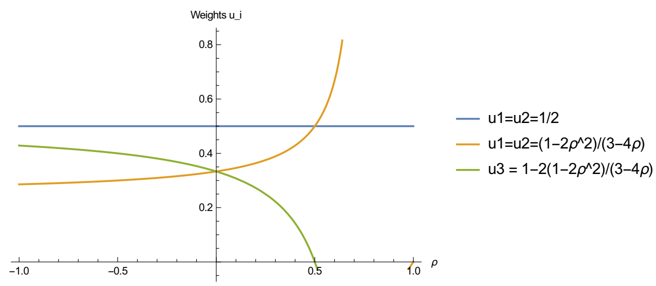

Figure 2. Asset weights for two cases $u_1 = u_2 = 1/2, u_3 = 0$ and $u_1 = u_2 = (1-\rho)/(3-4\rho), u_3 = (1-2\rho)/(3-4\rho)$. Since we do not allow short positions, we require $u_i \geq 0$.

The second solution reads:

$$u_1 = \frac{1-\rho}{3-4\rho}, \tag{31}$$
$$u_2 = \frac{1-\rho}{3-4\rho}, \tag{32}$$
$$u_3 = \frac{1-2\rho}{3-4\rho}, \tag{33}$$
$$\lambda = 0, \tag{34}$$
$$s_2 = \sigma_P^2 = \frac{1-\rho^2}{3-4\rho}. \tag{35}$$

From equation (33) follows that whenever $\rho \geq 1/2$, $u_3 \leq 0$ (note that we do not allow short selling). Hence when correlations are significant, $\rho \geq 1/2$, adding a asset to the portfolio will increase it's volatility.

When correlation is weak $\rho \leq 1/2$, this is not the case, namely $s_2 \leq s_1$ when $\rho < 1/2$. This means that adding a third asset will decrease volatility of the portfolio.

This is also demonstrated in Fig. 1 and Fig. 2, where we present volatilities s_1, s_2 and weights for the two cases.

B. Case II: $\rho_0 \neq 0$

When first and the second asset are also correlated, results do not change significantly. However, they provide an explanation why, in practice, adding an asset to the portfolio reduces its variance.

Minimization procedure is the same as in the Case I. Again we obtain two results for which $u_1 > 0, u_2 > 0$:

$$u_1 = \frac{1}{2}, \quad (36)$$

$$u_2 = \frac{1}{2}, \quad (37)$$

$$u_3 = 0, \quad (38)$$

$$s_1 = \sigma_P^2 = \frac{1+\rho_0}{2}. \quad (39)$$

The second solution reads:

$$u_1 = \frac{1-\rho}{3-4\rho+\rho_0}, \quad (40)$$

$$u_2 = \frac{1-\rho}{3-4\rho+\rho_0}, \quad (41)$$

$$u_3 = \frac{1-2\rho+\rho_0}{3-4\rho+\rho_0}, \quad (42)$$

$$s_2 = \sigma_P^2 = \frac{1-\rho^2+\rho_0}{3-4\rho+\rho_0}. \quad (43)$$

From Eq. follows, that the first solution, the one with volatility s_1, minimizes the volatility (under the no-long position constraint) iff the correlation $\rho \geq (1+\rho_0)/2$.

In practice correlations are typically finite. Hence when the correlation ρ is larger than $\rho > (1+\rho_0)/2$, adding an asset to the portfolio will increase the portfolio volatility.

Most realistic correlations are medium, hence adding an asset typically will decrease the portfolio volatility.